EPQ model with imperfect quality items of raw material and finished product

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ABSTRACT

The purpose of this paper is to develop an economic production model (EPQ) with imperfect quality items of raw material. The raw material needed for the production of the finished product is acquired from a supplier at the beginning of the inventory cycle. It is assumed that the raw material contains a percentage of imperfect quality items. A 100% screening process for detecting the imperfect quality items is conducted. In this model both perfect and imperfect quality items are used in the production process. This yields two types of finished products with perfect and imperfect quality. Both types of finished products are assumed to have continuous demand. Two mathematical models are formulated for two possible cases that may be encountered. The optimal production quantity is determined, and numerical examples illustrating the models are given.

Keywords: economic production model, raw material, perfect and imperfect quality.

INTRODUCTION

The classical inventory models ignore many factors encountered in real life situations. For instance, the classical EPQ model does not take into consideration the raw material used in production and assumes that all items produced are of the same quality. Similarly, the classical economic order quantity model (EOQ) assumes that items acquired by a retailer or a manufacturer are all of perfect quality. This is not always the case in real life.

Recently, numerous research papers have been published on EPQ/EOQ models with imperfect quality items. This was triggered by (Salameh & Jaber, 2000). They developed an EOQ model to determine the optimal lot size where each lot delivered by the supplier contains imperfect items with a known probability density function. An EPQ model with the reworking of imperfect quality items was studied by (Hayek & Salameh, 2001). (Chiu, 2003) considered an EPQ model with random defective rate, a reworking process, and backlogging. (Ozdemir, 2007) examined an EOQ model with defective items where shortages are backordered. (Salameh & El-Kassar, 2007) presented an EPQ model that accounts for the cost of raw material. (El-Kassar et al., 2008) studied an economic production quantity model where the items produced are of two different qualities. (El-Kassar, 2008) examined an EOQ model with imperfect quality items where the imperfect quality items are sold at a discounted price and the demands for both perfect and imperfect quality items are continuous during the inventory cycle. (El-Kassar et al., 2012) presented an EPQ model with imperfect quality raw material needed for production. Two scenarios were considered. In the first, the imperfect quality items of the raw material are sold at a discounted price at the end of the screening period. In the second scenario, the imperfect quality items are kept in stock until the end of the inventory cycle and returned to the supplier when the next order is received. (Khan et al., 2011) presented an extensive review of EOQ/EPQ models with imperfect quality items.

In this paper, we examine an EPQ model that takes into consideration the raw material needed for the production of the finished product. The raw material is acquired from a supplier at the beginning of the inventory cycle. It is assumed that the raw material contains a percentage of imperfect quality items. A 100% screening process for detecting the imperfect quality items of raw material is conducted at a rate greater than the production rate. The perfect and imperfect quality items screened are used in the production process. The perfect quality items of raw material used in production yield perfect quality finished items. On the other hand, when the imperfect quality items of raw material are used, the finished items produced are of imperfect quality. Both types of finished products are assumed to have continuous demand. Two cases are considered depending on which of the two types of the finished product is sold out first. The rest of this paper is organized as follows. The mathematical models for the two cases are developed in section 2. In section 3, a numerical example is given to illustrate the model and to compare the optimal polices of the two cases. Section 4 contains a conclusion and future research suggestions.

THE MATHEMATICAL MODEL

The following notations are used to develop the proposed model

D_p = demand rate of perfect quality items of raw material
D_i = demand rate of imperfect quality items of raw material
P = production rate ($P<x$)
y = raw material ordered size
q = percentage of imperfect quality items of raw material
$1-q$ = percentage of perfect quality items of raw material
x = screening rate for imperfect quality items of raw material
y_{imax} = maximum inventory level of imperfect finished items
y_{pmax} = maximum inventory level of perfect finished items
K_p = set up cost for production
Ks = ordering cost from supplier
h_r = raw material holding cost per unit per unit time
h_p = production holding cost per unit per unit time
C_r = unit cost of raw material
S_p = unit selling price of perfect quality items
S_i = discounted unit selling price of imperfect quality items
S_d = unit selling discounted price of imperfect quality items
T = inventory cycle length
T_i = inventory cycle length of imperfect quality items
T_P = inventory cycle length of perfect quality items
t_s = screening period
t_{1i} = production period for imperfect quality finished items
t_{1p} = production period for imperfect quality items

Consider a production model where the production rate is P. The raw material used in the production process is ordered at the beginning of the production period from a supplier. An order of size y of raw material is acquired at the beginning of the production cycle. Each batch of raw material received is assumed to contain a known percentage q of imperfect quality items. The items are screened at the beginning of the inventory cycle. Both types of items of raw material are used in the production process. The perfect quality items of raw material screened, $y(1-q)$ units, are used to produce perfect quality finished product. The remaining imperfect quality items, yq, yield imperfect quality finished product. The demand rates for the perfect and imperfect quality items produced are D_p and D_i, respectively.

To develop the model, the following assumptions are made:

1. The demand rates, D_p and D_i, are known and constant.
2. The percentage of imperfect quality items of raw material q is known and constant.
3. The production rates for perfect and imperfect quality finished items are assumed to be larger than their corresponding demand rates.
4. Raw material is ordered and received instantaneously at the beginning of the production period.
5. One item of raw material is needed to produce one item of the finished product.

The order of size y of raw material contains qy imperfect quality items. These items are processed at a rate P to produce the imperfect quality finished product. Therefore, the length of the production period for the imperfect quality finished items is

$$t_{1i} = \frac{qy}{P}. \tag{1}$$

Similarly, the $(1-q)y$ perfect quality items of raw material are also processed at the rate P so that the length of the production period for perfect quality finished items is

$$t_{1p} = \frac{(1-q)y}{P}. \tag{2}$$

The finished items produced are used to meet the demand. Since the demand rates are D_p and D_i, for perfect and imperfect quality finished items, respectively, we have that the length of the inventory cycle for the imperfect quality finished items is

$$T_i = \frac{qy}{D_i}, \tag{3}$$

and that of perfect quality is

$$T_p = \frac{(1-q)y}{D_p}. \tag{4}$$

The combined inventory cycle length T for both types of finished items depends on the larger value among T_i and T_P. At the end of the production period, inventory is depleted at a rate of D until items of either type are out of stock. That is, $T = \max\{T_i, T_p\}$. Two cases are considered: $T_i \le T_p$, and $T_i > T_p$. In the first case, using (3) and (4), we have that the condition $T_i \le T_p$ is equivalent to $\frac{D_i}{D_p} \ge \frac{q}{1-q}$. Similarly, the condition for the second case is $\frac{D_p}{D_i} \ge \frac{1-q}{q}$.

Suppose that $T_i < T_P$. Then, $T = T_P$ and the inventory levels for raw materials and the finished product are shown in figures 1, 2, 3 and 4. At the beginning of the production cycle, a 100% screening process for detecting the imperfect quality items is conducted at a screening rate x, where $x > P$. Throughout the screening period, the perfect and imperfect quality items of raw material are used in the production process. The behavior of the inventory level of perfect quality items of raw material is shown in figure 1.

At the beginning of the inventory period, this level starts at $(1-q)y$ and is depleted at a rate P until the end of this production period where it reaches zero. See figure1. Similarly, the inventory level of imperfect quality items of raw material starts at qy and decreases at a rate P, see figure 2.

During the production period, finished items of perfect quality are produced at a rate P. Part of these items is sold at a rate D_P. Therefore, an inventory of perfect quality finished items is accumulating throughout the production period at a rate $P-D_P$ until a maximum level of y_{pmax} is reached, see figure 3. From (2), this maximum level is

$$y_{p\max} = (1-q)\left(1-\frac{D_p}{P}\right)y. \tag{5}$$

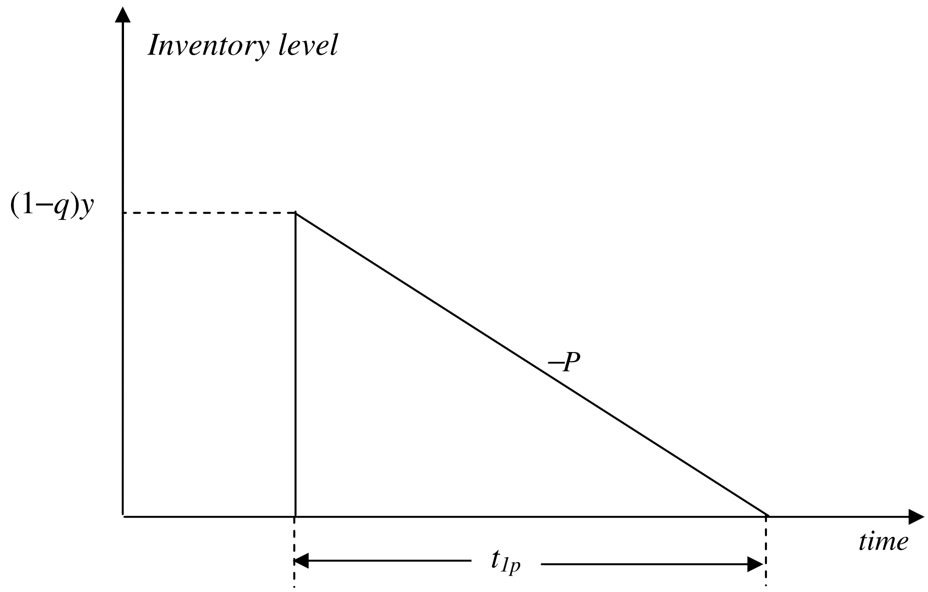

Figure1: Perfect items of raw material

A similar argument gives that the maximum level of imperfect finished items is

$$y_{i\max} = qy\left(1-\frac{D_i}{P}\right) \tag{6}$$

To find the optimal order quantity, we first calculate the total cost per inventory cycle which is the sum of purchasing cost, production cost, ordering cost of perfect and imperfect raw material, setup cost of production, raw material holding cost and finished product holding cost. Except for the holding costs, the components are as follows:

Purchasing cost of raw material	$= C_r y,$	(7)
Production cost	$= C_p y,$	(8)
Ordering cost of raw material	$= K_r,$	(9)
Setup cost	$= K_p.$	(10)

The holding cost of perfect quality items of raw materials is the product of the average inventory, cycle length, and the holding cost per unit per unit time h_r. To find the average inventory, the area under the curve is divided by the total cycle length. The area under the curve representing the inventory level of perfect quality raw material, see figure 1, is given by

$$\text{Area} = \frac{(1-q)\,yt_{1p}}{2}. \tag{11}$$

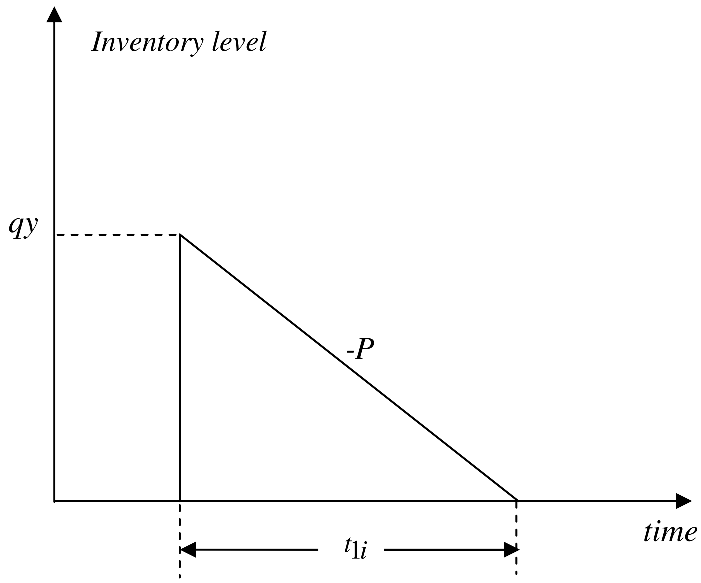

Figure2: Perfect items of raw materials

From (2), the area in (11) becomes

$$\text{Area} = \frac{(1-q)^2 y^2}{2P}. \tag{12}$$

Using (12), the holding cost of perfect raw material becomes

$$\text{Holding cost of perfect raw material} = \frac{(1-q)^2 y^2}{2P} h_r. \tag{13}$$

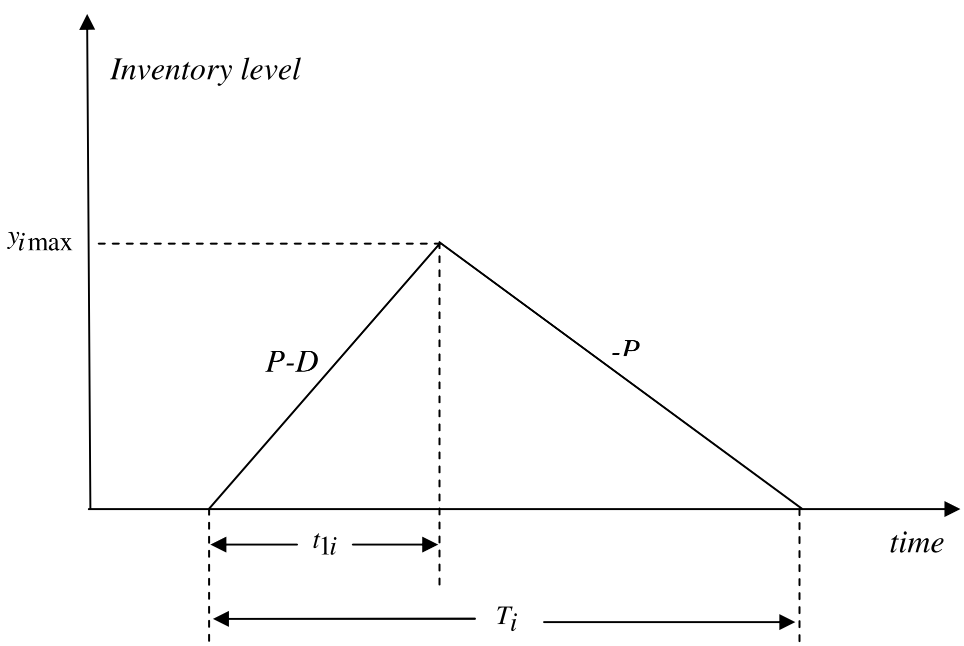

Figure3: Finished Imperfect Product Inventory Level

Similarly, the holding cost of imperfect raw material, see figure 2, is

$$\text{Holding cost of imperfect raw material}=\frac{q^2y^2}{2P}h_r. \tag{14}$$

The holding cost for perfect quality finished items is the product of the average inventory, cycle length and the sum of the two holding cost per unit per unit time, h_r and h_p. Since the area under the curve representing the inventory level of perfect finished items is $(T_p y_{p\max})/2$, we have

$$\text{Average inventory}=\frac{y_{p\max}}{2}T_p\frac{1}{T}=\frac{y_{p\max}}{2}. \tag{15}$$

Using (3) and (6) we obtain

$$\text{Holding cost for perfect finished items}=\frac{y^2(1-q)^2}{2D_p}\left(1-\frac{D_p}{P}\right)(h_p+h_r). \tag{16}$$

Similarly, by using (3) and (6), we obtain

$$\text{Holding cost for imperfect finished items}=\frac{q^2y^2}{2D_i}\left(1-\frac{D_i}{P}\right)(h_p+h_r). \tag{17}$$

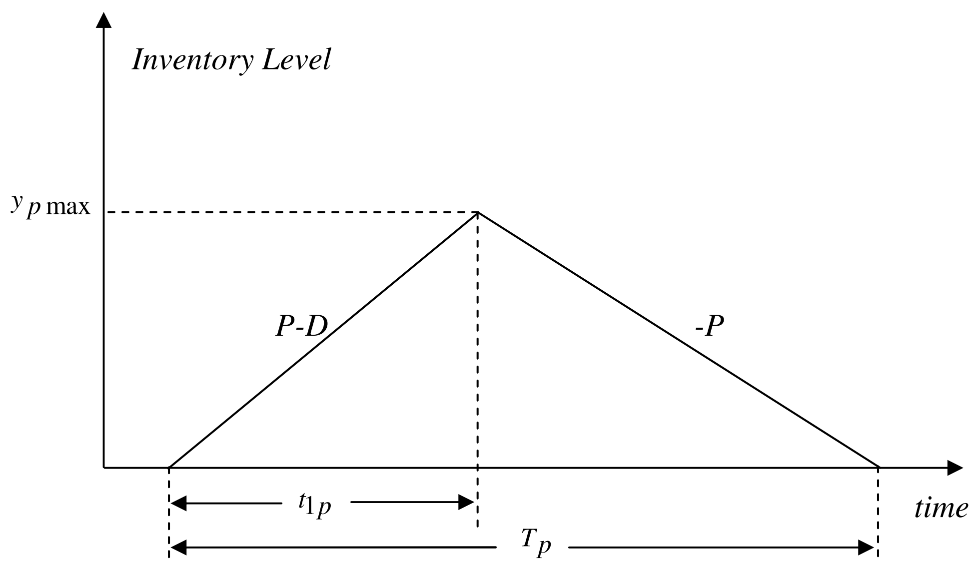

Figure 4: Finished Perfect Product Inventory Level

Now, the total inventory cost per cycle function $TC(y)$ is the sum of (7)-(10), (13),(14), (16) and (17). That is,

$$\begin{aligned}TC(y)&=C_r y+C_p y+K_r+K_p+\frac{y^2}{2P}\left(q^2+(1-q)^2\right)h_r\\&+\frac{q^2y^2}{2D_i}\left(1-\frac{D_i}{P}\right)(h_p+h_r)+\frac{(1-q)^2y^2}{2D_p}\left(1-\frac{D_p}{P}\right)(h_p+h_r).\end{aligned} \tag{18}$$

The total inventory cost per unit time function $TCU(y)$ is obtained by dividing (18) by the inventory cycle length $T = T_p = \dfrac{(1-q)y}{D_p}$. Hence,

$$TCU(y) = \frac{(C_r + C_p)D_p}{1-q} + \frac{(K_r + K_p)D_p}{(1-q)}\frac{1}{y} + \frac{q^2 + (1-q)^2}{2(1-q)P}D_p h_r y$$

$$+\left[\frac{q^2}{2(1-q)}\left(1-\frac{D_i}{P}\right)\frac{D_p}{D_i} + \frac{(1-q)}{2}\left(1-\frac{D_p}{P}\right)\right](h_p + h_r)y. \tag{19}$$

Now, the total revenue function $TR(y)$ is the sum of the sales revenue of the finished product and the discounted sales of the imperfect quality items. That is,

$$TR(y) = S_p(1-q)y + S_i\, q\, y. \tag{20}$$

Dividing (20) by T, we have that the total revenue per unit time is

$$TRU(y) = S_p D_p + S_i\ D_p \frac{q}{1-q}. \tag{21}$$

The economic order quantity is determined by maximizing the total profit. The total profit function per cycle is $TP(y) = TR(y) - TC(y)$. From (19) and (21), we get

$$TPU(y) = S_p D_p + S_i\, q\, D_p - (C_r + C_p)\frac{D_p}{(1-q)} - (K_r + K_p)\frac{D_p}{(1-q)y}$$

$$-\frac{q^2 y}{2(1-q)}\left(1-\frac{D_i}{P}\right)(h_p + h_r)\frac{D_p}{D_i} - \frac{(1-q)y}{2}\left(1-\frac{D_p}{P}\right)(h_p + h_r) \tag{22}$$

$$-\frac{y}{2(1-q)P}[q^2 + (1-q)^2]\, D_p h_r.$$

Differentiating the expression of $TPU(y)$ in (22) with respect to y, we get

$$\frac{d}{dy}(TPU(y)) = \frac{(K_r + K_p)D_p}{(1-q)y^2} - \frac{[q^2 + (1-q)^2]}{2(1-q)P}D_p h_r$$

$$-\frac{q^2}{2(1-q)}\left(1-\frac{D_i}{P}\right)(h_p + h_r)\frac{D_p}{D_i} - \frac{(1-q)}{2}\left(1-\frac{D_p}{P}\right)(h_p + h_r). \tag{23}$$

Setting the derivative in (23) equal to zero and solving for y, we obtain the economic order quantity

$$y^* = \sqrt{\frac{2P(K_r + K_p)D_p}{\left(q^2 + (1-q)^2\right)D_p h_r + (h_p + h_r)\left(q^2(P - D_i)\dfrac{D_p}{D_i} + (1-q)^2(P - D_p)\right)}}. \tag{24}$$

Since the second derivative of $TPU(y)$ is

$$\frac{d^2}{dy^2}(TPU(y)) = -\frac{2(K_r + K_p)D_p}{(1-q)y^3}, \tag{25}$$

and the right hand side of (25) is always negative, the optimal order quantity y^* is the unique maximizer of the $TPU(y)$ function.

Now we develop the mathematical model for the case when $T_i > T_p$ so that $T = \max(T_i, T_p) = T_i$. In this case, we assume that the remaining imperfect quality finished items are sold in one batch at a low discounted price at time T_p. The behavior of the inventory level of raw material is shown in figure 5.

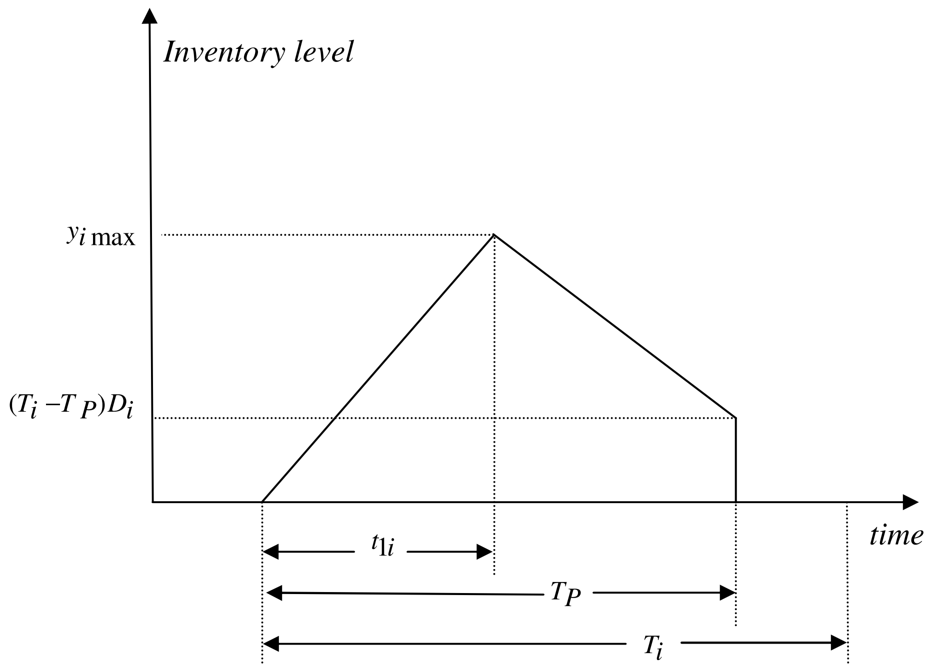

Figure 5: Finished Product of imperfect Items

To find the optimal order quantity, we calculate the total cost per cycle which is identical to that of the previous case except for the holding cost of finished product of imperfect items. The area under the curve representing the inventory level in figure 5 is

$$\text{Area} = \frac{-q^2y^2}{2P} + \frac{q(1-q)y^2}{D_p} - \frac{(1-q)^2y^2}{2(D_p)^2}D_i. \tag{25}$$

Dividing (26) by T, we get the average inventory. Then multiplying the average inventory by $(h_r + h_p)$ and T, we obtain the holding cost for finished imperfect items as

$$\text{Imperfect finished items holding cost} = \left\{-\frac{q^2}{p} + \frac{2q(1-q)}{D_p} - \frac{(1-q)^2D_i}{(D_p)^2}\right\}\frac{y^2}{2}(h_r + h_p). \quad (27)$$

Replacing the holding cost for finished imperfect items term in (17) by (27), we obtain

$$TC(y) = (C_r + C_p)y + K_r + K_p + \frac{y^2}{2P}\left(q^2 + (1-q)^2\right)h_r + \left[\frac{(1-q)^2}{D_p}(1-\frac{D_p}{P}) - \frac{q^2}{P} + \frac{2q(1-q)}{D_p} - \frac{(1-q)^2 D_i}{(D_p)^2}\right]\frac{y^2(h_p + h_r)}{2}. \tag{28}$$

Dividing (28) by $T = T_i = \dfrac{q\,y}{D_i}$, we obtain the TCU (y) function for the second case as

$$TCU(y) = (C_r + C_p)\frac{D_i}{q} + \frac{(K_r + K_p)D_i}{q}\frac{1}{y} + \frac{D_i\left(q^2 + (1-q)^2\right)h_r}{2Pq}y + \left[\frac{(1-q)^2}{qD_p}(1-\frac{D_p}{P}) - \frac{q}{P} + \frac{2(1-q)}{D_p} - \frac{(1-q)^2}{q(D_p)^2}D_i\right]\frac{D_i\,(h_p + h_r)}{2}y. \tag{29}$$

Now, the total revenue function $TR(y)$ is

$$TR(y) = S_p(1-q)y + S_i(yq - (T_i - T_P)D_i) + S_d(T_i - T_p)D_i. \tag{30}$$

Dividing (30) by $T = T_i = \dfrac{q\,y}{D_i}$, we have that the total revenue per unit time is

$$TRU(y) = \frac{S_p(1-q)}{q}D_i + \left(S_d - S_i\right)\left(1 - \frac{(1-q)^2 D_i^2}{qD_p}\right) \tag{31}$$

Maximizing the total profit per unit time function in a manner similar to that of the first case, we get the economic order quantity

$$y^* = \sqrt{\frac{2P(K_r + K_p)}{\left(q^2 + (1-q)^2\right)h_r + (h_p + h_r)\left[-q^2 + \frac{2Pq(1-q)}{D_p} + \frac{(1-q)^2}{D_p}\left(P - D_p - \frac{PD_i}{D_p}\right)\right]}}. \tag{32}$$

NUMERICAL EXAMPLE

Consider a production process where the production rate is 100 items per day. The production process uses raw material with imperfect quality items. The percentage of imperfect quality items of raw material is 30%. Screening for imperfect quality items of the raw material is conducted at a rate 200 items per day. The perfect and imperfect quality items of raw material are used in the production process resulting in a finished product with perfect and imperfect quality items. The daily demand rates for both types of finished items are 30 and 20 units, respectively. The ordering cost for the raw material is \$104.5 and the setup cost is \$150. The holding cost per unit of raw material per day is \$0.01 while the holding cost due to production is \$0.02 per unit per day. Hence, the holding cost of one unit of the finished product is \$0.03 per day. The purchasing cost of one item of raw material is \$5 and the unit production cost is \$10. The unit selling prices per unit are S_i = \$25 and S_p = \$30.

The parameters of the problem are P = 100, q = 0.3, x = 200, D_P = 30, D_i = 20, K_r = 104.5, K_p = 150, h_r = 0.01, h_p = 0.02, C_r = 5, C_p = 10, S_i = 25 and S_p = 30. From (24), the optimal number of items produced is y^* = 1000. The optimal number of perfect items produced during the production cycle is 700 units and that for the imperfect items is 300 units. The length of the inventory period for imperfect items is 15 days and the length of inventory period for perfect items is 23.33 days. The production period for imperfect items is 3 days and that for imperfect items is 7 days. The total inventory cost per day is \$664.67, the total revenue per day is \$1221.43, and the maximum total profit per day is \$556.75.

Now suppose that the daily demand rate for both types of finished items are D_P = 60 and D_i = 20. Then, the ratio $D_p / D_i = 3 > (1-q)/q = 2.33$. Hence, $T_i > T_p$ and we have the second case. Suppose that the unit selling discounted price for the imperfect quality items sold at the end of the inventory cycle when the perfect quality items sold out is S_d =\$15. From (32), the optimal number of items produced is y^*=1400. The number of perfect quality finished items produced during the production cycle is 980 units and that for the imperfect items is 420 units. The length of inventory period for imperfect items is 21 days and the length of inventory period for perfect items is 16.33 days. The production period for imperfect items is 4.2 days and that for perfect items 9.8 days. The total inventory cost per day is \$1024.13, the total revenue per day is \$2054.65, and the maximum total profit per day is \$1021.55.

CONCLUSION

The model presented in this paper extends the classic economic production quantity (EPQ) model to the case where raw material with imperfect quality items is used in the production process. The finished items produced are also of perfect and imperfect quality. Two cases for this model were considered. The optimal operating policy was derived by maximizing the total profit per unit time. Explicit expressions for the optimal order quantity for the two cases were obtained. The uniqueness of the optimal solutions was demonstrated. Numerical examples were given to illustrate how the optimal policy can be determined.

For future research, the effect of time value of money and credit facility for this model may be investigated. The model can be extended by considering probabilistic percentage of imperfect quality items.

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